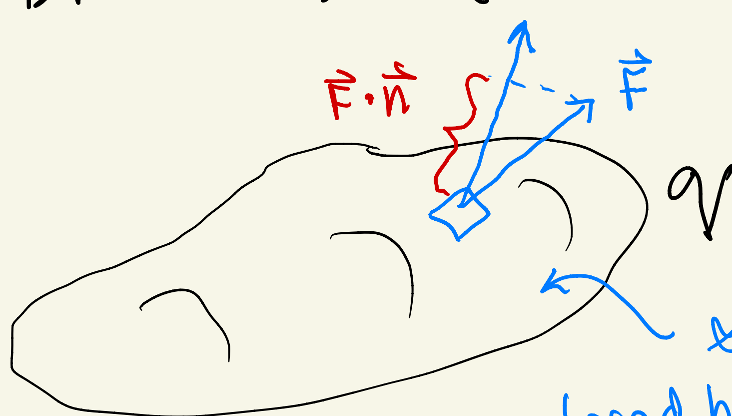
§ 16.9 Divergence Theorem:

Statement:

$$\underbrace{\iiint_{\mathcal{V}} \mathrm{Div}\,\vec{F}\, dV}_{\text{The Ch 15 triple integral of Div } \vec{F} \text{ over a 3-d vol } \mathcal{V}} = \underbrace{\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, dS}_{\text{The flux of } \vec{F} \text{ through the boundary } \mathcal{S} \text{ of } \mathcal{V}}$$

$\vec{F} = \overrightarrow{(M, N, P)}$ $\quad \mathrm{Div}\,\vec{F} = M_x + N_y + P_z$

Picture:



In words: "The integral of Div $\vec{F}$ over a volume $\mathcal{V}$ is always equal to the flux of $\vec{F}$ through the boundary"

Recall: In the fluid model $\vec{F} = \delta\vec{v}$ = mass flux vector, $\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, dS = \dfrac{\text{mass}}{\text{time}}$ outward thru $\mathcal{S}$

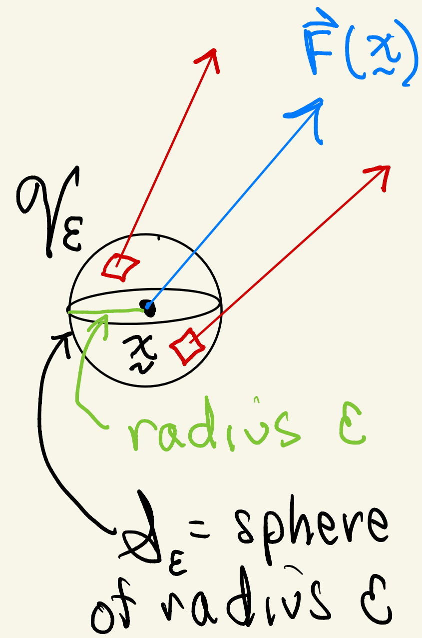
Example: Use the Divergence theorem to give a physical interpretation to Div $\vec{F}$.

That is, we can compute Div $\vec{F}$ at each point $P = \underset{\sim}{x} = (x, y, z)$ as $\text{Div}\,\vec{F}(\underset{\sim}{x}) = \underbrace{\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}}_{\text{A number computed at each point } \underset{\sim}{x}}$

Q: What does Div $\vec{F}$ measure?

Ans: Div $\vec{F}$ measures "Flux per Volume"

To see this, take a small ball $\mathcal{V}_\varepsilon$ of radius ε centered at $\underset{\sim}{x}$.



Apply the Divergence Thm:

$$\iiint_{\mathcal{V}_\varepsilon} \text{Div}\,\vec{F}\,dV = \iint_{\mathcal{S}_\varepsilon} \vec{F}\cdot\vec{n}\,dS$$

The trick: If ε is suffiently small, and $\vec{F}$ and its derivatives are continuous, then the value of Div $\vec{F} \approx \text{Div}\,\vec{F}(\underset{\sim}{x})$ throughout $\mathcal{V}_\varepsilon$

That is:

$$\iiint_{V_\varepsilon} \operatorname{Div}\vec{F}\,dv = \iiint_{V_\varepsilon} \operatorname{Div}\vec{F}(\underset{\sim}{x})\,dV + \text{error}$$

$\uparrow$ center point

$\uparrow$ smaller than $|V_\varepsilon| = \text{Vol } V_\varepsilon$

$$= \operatorname{Div}\vec{F}(\underset{\sim}{x}) \underbrace{\iiint_{V_\varepsilon} dv}_{|V_\varepsilon| = \text{vol of } V_\varepsilon} + \text{error}$$

$$= |V_\varepsilon| \operatorname{Div}\vec{F}(\underset{\sim}{x}) + \text{error}.$$

Solving for $\operatorname{Div}\vec{F}(\underset{\sim}{x})$ gives:

$$\operatorname{Div}\vec{F}(\underset{\sim}{x}) = \frac{1}{|V_\varepsilon|} \iint_{S_\varepsilon} \vec{F}\cdot\vec{n}\,dS + \underbrace{\frac{\text{error}}{|V_\varepsilon|}}_{\text{tends to 0 as } \varepsilon \to 0}$$

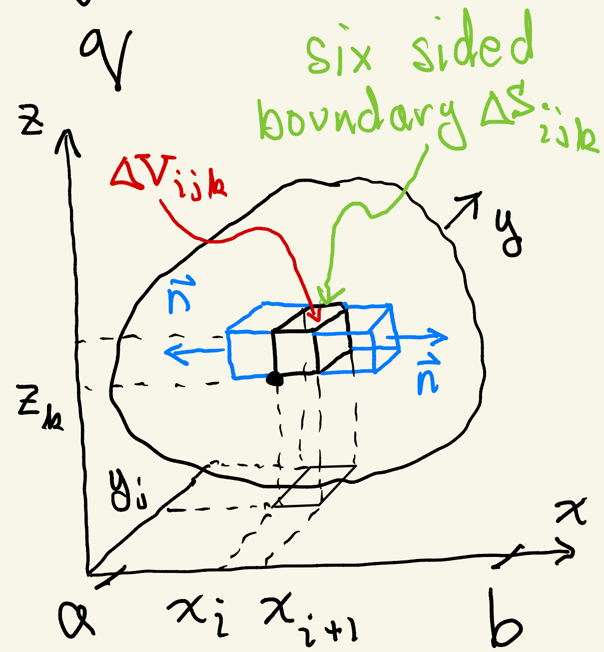
Conclude: $\operatorname{Div}\vec{F}(\underset{\sim}{x}) = \lim_{\varepsilon \to 0} \frac{1}{|V_\varepsilon|} \iint_{S_\varepsilon} \vec{F}\cdot\vec{n}\,dS$

$\doteq$ "Flux of $\vec{F}$ per Volume"

Note: This works for any volume $V_\varepsilon = \underbrace{\varepsilon V_0}_{\text{so it scales w } \varepsilon}$

Example: The fact that $\text{Div}\,\vec{F} = \frac{\text{Flux}}{\text{Vol}}$ explains why the Divergence Theorem is true:

I.e. Approximate triple integral $\iiint_{\mathcal{V}} \text{Div}\,\vec{F}\,dV$ as a Riemann Sum:



$$\iiint_{\mathcal{V}} \text{Div}\,\vec{F}\,dV \approx \sum_{i,j,k} \text{Div}\,\vec{F}_{ijk}\, \Delta V$$

$$\approx \sum_{ijk} \frac{\iiint_{\Delta S_{ijk}} \vec{F}\cdot\vec{n}\,dS}{\cancel{\Delta V}}\,\cancel{\Delta V}$$

$$= \sum_{ijk} \iiint_{\Delta S_{ijk}} \vec{F}\cdot\vec{n}\,dS$$

The flux integral cancels on all shared sides as outer normal switches sign!

$$\approx \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS$$

only the outer sides on the boundary of $\mathcal{V}$ have no adjacent boundary to cancel them!

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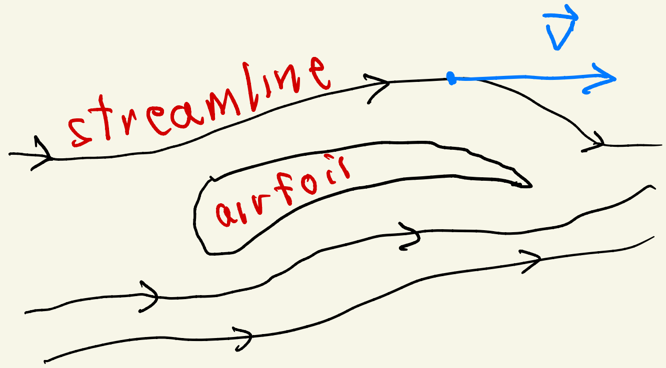
Example: The most important application of the Divergence Theorem –

Assume $\vec{F} = \delta \vec{v}$ in the fluid model where

$\delta(x,y,z) = \frac{\text{mass}}{\text{vol}}$

and

$\vec{v}(x,y,z) = \text{velocity}$



Q: What constraint must δ and $\vec{v}$ satisfy to ensure that mass is conserved?

Ans: $\boxed{\text{Div}\, \delta \vec{v} = 0}$ Intuitively, $\text{Div}\, \delta\vec{v}(\underset{\sim}{x}) = \frac{\text{Flux}}{\text{vol}}$ at $\underset{\sim}{x}$, so if $\text{Div}\, \delta \vec{v}(\underset{\sim}{x}) \neq 0$, mass is either created or destroyed at $\underset{\sim}{x}$. We get a careful argument by Div Thm

The idea – if mass is conserved in every volume $\mathcal{V}$, then we must have that flux of mass thru the boundary of $\mathcal{V}$ must $= 0$ ... otherwise mass would be accumulating in $\mathcal{V}$.

So Conservation of Mass $\Rightarrow \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = 0$ for all $\mathcal{V}$

Defn: We say conservation of mass holds if $\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 0$ for $\mathcal{S}$ the boundary of any 3-d surface $\mathcal{V}$. Now apply Div. Thm

Div Thm: $$\iiint_{\mathcal{V}} \operatorname{Div}\vec{F}\,dV = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 0$$

This implies $\operatorname{Div}\vec{F} = 0$. If $\operatorname{div}\vec{F}(\underset{\sim}{x}) \neq 0$ at a point, then choose a small volume $\mathcal{V}_\varepsilon$ in which $\operatorname{div}\vec{F}$ has same sign as $\operatorname{Div}\vec{F}(\underset{\sim}{x})$, say positive, then

$$\iiint_{\mathcal{V}_\varepsilon} \underbrace{\operatorname{Div}\vec{F}}_{\text{pos}}\,d\bar{V} > 0 \quad \#$$

Conclude: the condition for Conservation of Mass is:

$$\boxed{\operatorname{Div} \delta\vec{V} = 0}$$

Continuity Equation

- For time dependent flows, same idea

$$\delta(x,y,z,t), \quad \vec{v}(x,y,z,t)$$

Then Conservation of mass is

$$\underbrace{\delta_t + \mathrm{Div}(\delta \vec{v}) = 0}_{\mathrm{Div}_{t,\underset{\sim}{x}}(\delta,\, \delta\vec{v}) = 0} \qquad \text{Continuity Equation} \qquad (*)$$

The continuity equation is the first partial differential equation of fluid mechanics. For example, the continuum version of Newton's Laws is expresses conservation of mass (*) together with an equation for conservation of momentum and conservation of energy, both of which can be derived from the Divergence Theorem. Note that (*) is NONLINEAR! The equations for Fluids are "Fiercely Nonlinear"